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Irreducible representations of the Poincaré parasuperalgebra†

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Abstract. We explicitly describe all the irreducible unitary representations of the Poincaré parasuperalgebra, i.e. the parasupersymmetric extension of the Lie algebra of the Poincaré group. This parasuperalgebra includes, as a particular case, the usual Poincaré superalgebra and can serve as the group-theoretical foundation of parasupersymmetric quantum field theory.

1. Introduction

About 20 years ago there appeared a new symmetry principle in physics which supposed the existence of symmetry transformations mixing bosonic and fermionic states [1, 2]. In addition to the usual Poincaré group generators the supplementary fermionic generators, which connect fields with different statistics, were taken into consideration. Supersymmetry provides a mechanism for the cancellation of the ultraviolet divergences in quantum field theory. It also makes it possible to unify spacetime symmetries (i.e. Poincaré invariance) with internal symmetries [3] and opens additional ways for the search for unified field theories, including all the types of interactions [2].

Supersymmetric quantum field theory (SSQFT) [4] induced the appearance of supersymmetric quantum mechanics (SSQM) [5]. While being very interesting in its own right as a relative simple mathematical model of a physical system with supersymmetry, SSQM stimulated a deeper understanding of ordinary quantum mechanics and provided new ways to solve some problems using, e.g., the concept of partner superpotential [6].

SSQM in its turn has been generalized [7] to parasupersymmetric quantum mechanics (PSSQM). The latter deals with bosons and $p = 2$ parafermions having parastatistical properties [8].

The independent version of PSSQM corresponding to positive defined Hamiltonians was proposed in [9], the theories intermediate between SSQM and PSSQM have been discussed in [10].

A more recent theory called PSSQM has awoken interest and stimulated the appearance of a lot of articles, see [11] and references therein. Parasuperpotentials admitting Lie and non-Lie [12] symmetries were investigated in [13], hidden $SU(3)$ symmetry of equations of PSSQM was established in [14].

The decisive step in the development of PSSQM was made by Beckers and Debergh [15] who asked for Poincaré invariance of the theory and formulated the group-theoretical foundations of so-called parasupersymmetric quantum field theory (PSSQFT). This theory is

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a natural generalization of SSQFT, dealing with parastatistics instead of the usual Fermi or Bose statistics and with the Poincaré parasupergroup (or Poincaré parasuperalgebra (PPSA)) instead of the Poincaré supergroup (or Poincaré superalgebra (PSA)). On the other hand, *this theory is a relativistic extension of the* PSSQM, *preserving the main properties of the* non-relativistic parasupercharges.

Moreover, some dynamical models were proposed in [15], which were parasupersymmetric analogues of the Wess–Zumino model [16].

We found it is necessary to analyse irreducible representations (IRs) of the PPSA for the following reasons:

(i) this is a way to establish the group-theoretical fundamentals of the PSSQFT;

(ii) it enables a new view to be generated—note the PSA which appears in our approach as a particular realization of the PPSA;

(iii) it indicates the specific role of the groups $SO(3)$, $SO(5)$ and $SO(2,3)$ in the construction of internal super- and parasupersymmetries;

(iv) finally, the description of these IRs is an interesting mathematical problem admitting an exact and elegant solution.

Using the Wigner induced representation method we find *all* the IRs of the PPSA, for time-like, light-like and space-like four-momenta. We also find covariant representations of the PPSA, which can have direct applications in PSSQFT.

2. The Poincaré parasuperalgebra

The Poincaré parasuperalgebra [15] includes ten generators P_ν, $J_{\nu\sigma}$ of the Poincaré group, satisfying the usual commutation relations

$$\begin{aligned}
&[P_\mu, P_\nu] = 0 \qquad [P_\mu, J_{\nu\sigma}] = \mathrm{i}(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu)\\
&[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho})\\
&J_{\mu\nu} = -J_{\nu\mu} \qquad \mu,\nu = 0,1,2,3
\end{aligned} \tag{2.1}$$

and four parasupercharges Q_A, $\bar{Q}_A$ $(A = 1,2)$ which satisfy the following double commutation relations

$$\begin{aligned}
&[Q_A,[Q_B,Q_C]] = [\bar{Q}_A,[\bar{Q}_B,\bar{Q}_C]] = 0\\
&[Q_A,[Q_B,\bar{Q}_C]] = -4Q_B(\sigma_\mu)_{AC}P^\mu\\
&[\bar{Q}_A,[Q_B,\bar{Q}_C]] = 4\bar{Q}_C(\sigma_\mu)_{BA}P^\mu.
\end{aligned} \tag{2.2}$$

Here σ_ν are the Pauli matrices, $(\cdot)_{AC}$ are the corresponding matrix elements.

Furthermore, parasupercharges commute with generators of the Poincaré group as Weyl spinors:

$$\begin{aligned}
&[J_{\mu\nu}, Q_A] = -\frac{1}{2\mathrm{i}}\left(\sigma_{\mu\nu}\right)_{AB} Q_B \qquad [P_\mu, Q_A] = 0\\
&[J_{\mu\nu}, \bar{Q}_A] = -\frac{1}{2\mathrm{i}}\left(\sigma_{\mu\nu}\right)^*_{AB} \bar{Q}_B \qquad [P_\mu, \bar{Q}_A] = 0
\end{aligned} \tag{2.3}$$

where $\sigma_{\nu\sigma} = -\sigma_{\sigma\nu} = \sigma_\nu\sigma_\sigma$.

The PPSA is a direct (and natural) generalization of the PSA [2]. Indeed, the PSA also includes 14 elements satisfying (2.1) and (2.3), but instead of (2.2) supercharges Q_A, $\bar{Q}_A$ satisfy the following anticommutation relations:

$$\begin{aligned}
&[Q_A, Q_B]_+ = Q_AQ_B + Q_BQ_A = 0 \qquad [\bar{Q}_A, \bar{Q}_B]_+ = 0\\
&[Q_A, \bar{Q}_B]_+ = 2(\sigma_\mu)_{AB}P^\mu.
\end{aligned} \tag{2.4}$$

We can ensure that (2.2) is a mere consequence of (2.4); however, the converse is not true. Thus, the PSA is a particular case of the more general algebraic structure called PPSA; in the same way that the usual Fermi statistics is a particular case of the parastatistics [8]. Moreover, by analogy with the PSA, P_σ and $J_{\sigma\lambda}$ are called even, but Q_A, $\bar{Q}_A$ are called odd elements of the PPSA.

Some representations of the PPSA were described in [15]. Here we present the complete description of all the IRs of the parasuperalgebra (2.1)–(2.3).

3. Casimir operators and classification of the IRs

To find the main Casimir operators of the PPSA it is convenient to introduce the following 4-vector [5]

$$B_\mu = W_\mu + X_\mu \tag{3.1}$$

where W_ν is the Lubanski–Pauli vector

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{3.2}$$

and X_ν are the following bilinear combinations of parasupercharges

$$\begin{aligned} X_0 &= \tfrac{1}{8}\{[Q_1, \bar{Q}_1] + [Q_2, \bar{Q}_2]\} \qquad & X_1 &= \tfrac{1}{8}\{[Q_1, \bar{Q}_2] + [Q_2, \bar{Q}_1]\} \\ X_2 &= \tfrac{1}{8}\{[Q_2, \bar{Q}_1] + [\bar{Q}_2, Q_1]\} \qquad & X_3 &= \tfrac{1}{8}\{[\bar{Q}_1, Q_1] + [Q_2, \bar{Q}_2]\}. \end{aligned} \tag{3.3}$$

Using (2.1)–(2.3) we find the following commutation relations

$$[B_\mu, P_\nu] = 0 \qquad [B_\mu, J_{\nu\sigma}] = \mathrm{i}(g_{\mu\nu} B_\sigma - g_{\mu\sigma} B_\nu) \tag{3.4a}$$

$$\begin{aligned} &[B_\mu, Q_A] = \tfrac{1}{2} P_\mu Q_A \qquad [B_\mu, \bar{Q}_A] = -\tfrac{1}{2} P_\mu \bar{Q}_A, \\ &[B_\mu, B_\nu] = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma} P^\rho B^\sigma \end{aligned} \tag{3.4b}$$

from which it follows that the operators

$$C_1 = P_\mu P^\mu \qquad C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2 \tag{3.5}$$

are the Casimir operators of the PPSA. Indeed, C_1 coincides with the usual Casimir operator of the Poincaré algebra commuting with Q_A, $\bar{Q}_A$ in accordance with (2.3). The second Casimir C_2 is essentially new and includes the Poincaré invariant operator $W_\nu W^\nu$ as a constituent part. Thus, an IR of the PPSA is, in general, reducible with respect to the Lie algebra of the Poincaré group.

We will search for representations of the algebra (2.1)–(2.3) in the momentum representations, thus the action of the displacement operators P_ν will reduce to multiplication by p_ν, $-\infty < p_\nu < \infty$. In this case, for any fixed p_ν, relations (2.2) and (3.4*b*) define the algebra of operators B_ν, Q_A and $\bar{Q}_A$ which is going to be the main object of our investigations.

As in the case of the ordinary Poincaré algebra [17, 19] we distinguish the three main classes of IRs corresponding the following values of C_1:

$$\text{(I)} \qquad P_\mu P^\mu = M^2 > 0 \tag{3.6a}$$

$$\text{(II)} \qquad P_\mu P^\mu = 0 \tag{3.6b}$$

$$\text{(III)} \qquad P_\mu P^\mu = -\eta^2 < 0. \tag{3.6c}$$

We will see that the IRs of the PPSA can be qualitatively distinguished for different values of C_1 as enumerated in (3.6*a*–*c*). Moreover, these classes can be subdivided in accordance with the different origins of eigenvalues of the second Casimir C_2 and of additional Casimir operators existing in particular classes I–III of IRs.

4. IRs of class I

If (3.6a) is valid then there exists the additional Casimir operator $C_3 = P_0/|P_0|$ the eigenvalues of which are $\varepsilon = \pm 1$. We restrict ourselves to considering IRs corresponding to $\varepsilon = +1$ (for the case $\varepsilon = -1$ refer to section 8). In this case we can define 'a Wigner little parasuperalgebra' (LPSA) associated with the time-like 4-vector $P = (M, 0, 0, 0)$. We set

$$B_k = W_k + X_k = -MS_k + X_k \equiv Mj_k \qquad k = 1, 2, 3 \tag{4.1}$$

and obtain from (3.4b)

$$[B_0, Q_A] = \tfrac{1}{2}MQ_A \qquad [B_0, \bar{Q}_A] = -\tfrac{1}{2}M\bar{Q}_A \tag{4.2}$$

$$[j_k, Q_A] = [j_k, \bar{Q}_A] = 0 \tag{4.3}$$

$$[j_k, j_j] = \mathrm{i}\varepsilon_{kjl}j_l. \tag{4.4}$$

On the other hand we obtain from (2.2)

$$[Q_A, [\bar{Q}_A, Q_B]] = 4MQ_B \qquad [\bar{Q}_A, [Q_A, \bar{Q}_B]] = 4M\bar{Q}_B \tag{4.5}$$

the other double commutators of Q_A and $\bar{Q}_A$ are equal to zero.

It follows from (4.5) that the relation (4.2) turns out to be an identity if (4.5) is satisfied.

In accordance with (4.3)–(4.5) the LPSA reduces to the direct sum of the Lie algebra the basis elements of which are j_a and the algebra of operators Q_A, $\bar{Q}_A$ characterized by the double commutation relations (4.5). Thus, to describe the IRs of this LPSA it is sufficient to find all the IRs of the subalgebras (4.4) and (4.5). Indeed, let $\tilde{j}_a$ and I_j be the basis elements of an IR of the algebra (4.4) and the unit operator in the space of this IR and Q'_A, $\bar{Q}'_A$ and I_Q are basis elements of an IR of the algebra (4.5) and the unit operator in the space of this IR. Then, setting

$$j_a = \tilde{j}_a \otimes I_Q \qquad Q_A = I_j \otimes Q'_A \qquad \bar{Q}_A = I_j \otimes \bar{Q}'_A \tag{4.6}$$

(where $\otimes$ denotes the direct (Kronecker) product) we come to the IR of the algebra (4.3)–(4.5). Moreover, such a correspondence is a homomorphism.

Relations (4.4) define the Lie algebra $AO(3)$ of the rotation group $O(3)$. IRs of this algebra are labelled by integers or half integers j so that

$$\tilde{j}_1^2 + \tilde{j}_2^2 + \tilde{j}_3^2 = j(j+1) \tag{4.7}$$

The corresponding basis elements $\tilde{j}_a$ are the square matrices of dimension $(2j+1) \times (2j+1)$ which can be chosen in the following form [18]:

$$\begin{aligned} (\tilde{j}_3)_{ab} &= \delta_{ab}(j+1-a) \qquad a, b = 1, 2, \ldots, 2j+1 \\ (\tilde{j}_1 \pm \mathrm{i}\tilde{j}_2)_{ab} &= \delta_{ab\pm 1}\sqrt{j(j+1) - (j-a+1)(j-a+1\pm 1)}. \end{aligned} \tag{4.8}$$

To find IRs of the algebra (4.5) we choose the new basis

$$\begin{aligned} Q_1 &= \sqrt{2M}(S_{51} + \mathrm{i}S_{52}) \qquad \bar{Q}_1 = \sqrt{2M}(S_{51} - \mathrm{i}S_{52}) \\ Q_2 &= \sqrt{2M}(S_{53} + \mathrm{i}S_{54}) \qquad \bar{Q}_2 = \sqrt{2M}(S_{53} - \mathrm{i}S_{54}) \end{aligned} \tag{4.9}$$

and use the following notations for commutators

$$\begin{aligned} &[Q_1, \bar{Q}_1] = 4MS_{12} \qquad [Q_2, \bar{Q}_2] = 4MS_{34} \\ &[Q_1, \bar{Q}_2] = 2M(\mathrm{i}S_{24} - \mathrm{i}S_{31} + S_{14} - S_{23}) \\ &[Q_2, \bar{Q}_1] = 2M(\mathrm{i}S_{31} - S_{23} + S_{14} - \mathrm{i}S_{24}) \\ &[Q_1, Q_2] = -2M(\mathrm{i}S_{31} + iS_{24} + S_{14} + S_{24}) \\ &[\bar{Q}_1, \bar{Q}_2] = 2M(-\mathrm{i}S_{31} - \mathrm{i}S_{24} + S_{14} + S_{23}). \end{aligned} \tag{4.10}$$

Formulae (4.9) and (4.10) are invertible, so that

$$
\begin{aligned}
S_{51} &= \frac{1}{2\sqrt{2M}}(Q_1+\bar{Q}_1) \qquad S_{52} = -\frac{\mathrm{i}}{2\sqrt{2M}}(Q_1-\bar{Q}_1)\\
S_{53} &= \frac{1}{2\sqrt{2M}}(Q_2+\bar{Q}_2) \qquad S_{54} = -\frac{\mathrm{i}}{2\sqrt{2M}}(Q_2-\bar{Q}_2)\\
S_{12} &= \frac{1}{4M}[Q_1,\bar{Q}_1] \qquad S_{34} = \frac{1}{4M}[Q_2,\bar{Q}_2]\\
S_{14} &= \frac{1}{8M}([Q_1,\bar{Q}_2]+[Q_2,\bar{Q}_1]+[\bar{Q}_1,\bar{Q}_2]-[Q_1,Q_2])\\
S_{23} &= \frac{1}{8M}([\bar{Q}_1,\bar{Q}_2]-[Q_1,Q_2]-[Q_1,\bar{Q}_2]-[Q_2,\bar{Q}_1])\\
S_{13} &= -\frac{\mathrm{i}}{8M}([Q_1,Q_2]+[\bar{Q}_1,Q_2]+[Q_1,\bar{Q}_2]+[\bar{Q}_1,\bar{Q}_2])\\
S_{24} &= \frac{\mathrm{i}}{8M}([Q_1,Q_2]-[Q_1,\bar{Q}_2]-[\bar{Q}_1,Q_2]+[\bar{Q}_1,\bar{Q}_2]).
\end{aligned}
\tag{4.11}
$$

Using (4.5) and (4.11) we immediately find the following commutation relations for $S_{kl}=-S_{lk},\ k,l=1,2,\dots 5$

$$[S_{kl},S_{mn}]=\mathrm{i}(\delta_{km}S_{ln}+\delta_{ln}S_{km}-\delta_{kn}S_{lm}-\delta_{lm}S_{kn}) \tag{4.12}$$

which characterize the Lie algebra $AO(5)$ of the rotation group in five-dimensional space.

IRs of the algebra $AO(5)$ are labelled by pairs of numbers (n_1,n_2) both integer or half integer, moreover $n_1 \geqslant n_2$ [18]. The corresponding basis elements are square matrices of dimension $N(n_1,n_2)$, where

$$N(n_1,n_2)=\tfrac{1}{6}(n_1-n_2+1)(n_1+n_2+2)(2n_1+3)(2n_2+1). \tag{4.13}$$

For the explicit form of these matrices see the appendix.

Thus we have proved that for $P_\nu P^\nu > 0$ the LPSA reduces to the direct sum of the algebras $AO(3)$ and $AO(5)$

$$LPSA = AO(3)\oplus AO(5) \tag{4.14}$$

It follows from the latter that IRs of the PPSA of class I with positive sign of energy are labelled by the sets of numbers (M,j,n_1,n_2). To find the explicit form of the corresponding basis elements of the PPSA we start with the exact form of the Lubanski–Pauli vector W'_ν in the frame of reference where $P=(M,0,0,0)$, which, in accordance with (3.2), (4.1) and (4.10), can be given by the following relations:

$$W'_0=0 \qquad W'_a=-M(j_a+\tfrac{1}{4}\varepsilon_{abc}S_{bc}+\tfrac{1}{2}S_{4a})\equiv -MS_a. \tag{4.15}$$

Here

$$j_a=\tilde{j}_a\otimes I_{N(n_1,n_2)} \qquad S_{kl}=I_{2j+1}\otimes\hat{S}_{kl} \tag{4.16}$$

$\tilde{j}_a$ and $\hat{S}_{kl}$ are basis elements of the IRs $D(j)$ and $D(n_1,n_2)$ of the algebras $AO(3)$ and $AO(5)$, correspondingly, $I_{N(n_1,n_2)}$ and I_{2j+1} are the unit matrices of dimensions $N(n_1,n_2)\times N(n_1,n_2)$ and $(2j+1)\times(2j+1)$.

The corresponding parasupercharges are present in (4.9). With the help of Lorentz transformation we find the explicit form of the Lubanski–Pauli vector and parasupercharges in an arbitrary frame of reference:

$$W_0=p_aS_a \qquad W_a=\varepsilon MS_a+\frac{p_aS_bp_b}{(E+M)} \tag{4.17}$$

$$
\begin{aligned}
Q_1 &= \frac{1}{\sqrt{E+M}}[(S_{51}+\mathrm{i}S_{52})(E+M+\varepsilon p_3)+\varepsilon(S_{53}+\mathrm{i}S_{54})(p_1-\mathrm{i}p_2)]\\
Q_2 &= \frac{1}{\sqrt{E+M}}[\varepsilon(S_{51}+\mathrm{i}S_{52})(p_1+\mathrm{i}p_2)+(S_{53}+\mathrm{i}S_{54})(E+M-\varepsilon p_3)]\\
\bar{Q}_A &= Q_A^+
\end{aligned}
\tag{4.18}
$$

where

$$E=\sqrt{M^2+p^2} \qquad p^2=p_1^2+p_2^2+p_3^2.$$

The explicit form of the generators of the Poincaré group, corresponding to the Lubanski–Pauli vector (4.17), is well known (see, e.g., [19]), and can be represented by the formulae

$$
\begin{aligned}
&P_0=\varepsilon E \qquad P_a=p_a\\
&J_{ab}=x_a p_b - x_b p_a + \varepsilon_{abc}S_c\\
&J_{0a}=x_0 p_a - \frac{\mathrm{i}\varepsilon}{2}\left[\frac{\partial}{\partial p_a},E\right]_+ - \varepsilon\frac{\varepsilon_{abc}p_b S_c}{E+M}.
\end{aligned}
\tag{4.19}
$$

Thus, we have enumerated all the non-equivalent IRs of the PPSA of class I and have found the explicit form of the corresponding basis elements, see (4.15), (4.18) and (4.19).

5. IRs of class II

In this case we again have the additional Casimir $C_3=P_0/|P_0|=\varepsilon=\pm 1$. As before we consider the case $\varepsilon=+1$, refer to section 8 for the other case.

To obtain the corresponding LPSA we choose the light-like 4-vector $P=(M,0,0,M)$. The corresponding algebra (2.2) reduces to the form

$$[Q_2,[\bar{Q}_2,Q_2]]=8MQ_2 \qquad [\bar{Q}_2,[Q_2,\bar{Q}_2]]=8M\bar{Q}_2 \tag{5.1}$$

$$[Q_2,[\bar{Q}_2,Q_1]]=8MQ_1 \qquad [\bar{Q}_2,[Q_2,\bar{Q}_1]]=8M\bar{Q}_1 \tag{5.2}$$

the remaining double commutators equal to zero.

Let us start with (5.1). Denoting

$$j_1=\frac{1}{4\sqrt{M}}(Q_2+\bar{Q}_2) \qquad j_2=\frac{\mathrm{i}}{4\sqrt{M}}(Q_2-\bar{Q}_2) \qquad j_3=\frac{1}{8M}[Q_2,\bar{Q}_2] \tag{5.3}$$

we find that j_a have to satisfy the relations (4.4), characterizing the algebra $AO(3)$. The relations (5.3) are invertible, thus the algebra (5.1) reduces to the algebra $AO(3)$. Then the relations (5.2) (completed by the zero double commutators) have only trivial solutions for Q_1 and $\bar{Q}_1$. So we come to the following general form of parasupercharges:

$$Q_2=2\sqrt{M}(j_1-\mathrm{i}j_2) \qquad \bar{Q}_2=2\sqrt{M}(j_1+\mathrm{i}j_2) \qquad Q_1=\bar{Q}_1\equiv 0 \tag{5.4}$$

where j_a are basis elements of the algebra $AO(3)$.

In accordance with (3.4*b*), (5.4) and (4.4) we obtain

$$
\begin{aligned}
&[B_0,Q_l]=\tfrac{1}{2}MQ_l \qquad [B_0,\bar{Q}_l]=-\tfrac{1}{2}M\bar{Q}_l \qquad B_3=B_0\\
&[B_0,B_1]=\mathrm{i}MB_2 \qquad [B_0,B_2]=-\mathrm{i}MB_1 \qquad [B_0,B_l]=0.
\end{aligned}
\tag{5.5}
$$

Defining

$$
\begin{aligned}
&B_0=W_0+X_0=W_0+Mj_3\equiv M(T_0-\tfrac{1}{2}(j-j_3))\\
&W_0=M(T_0-\tfrac{1}{2}(j+j_3)) \qquad B_1=W_1\equiv T_1 \qquad B_2=W_2\equiv T_2
\end{aligned}
\tag{5.6}
$$

we obtain from (5.5)

$$[T_0, T_1] = \mathrm{i}T_2 \qquad [T_0, T_2] = -\mathrm{i}T_1 \qquad [T_1, T_2] = 0 \tag{5.7}$$

$$[T_0, j_a] = [T_1, j_a] = [T_2, j_a] = 0. \tag{5.8}$$

We see that LPSA reduces to the direct sum of the algebras $AO(3)$ and $AE(2)$, characterized by relations (4.4) and (5.7) correspondingly. In other words

$$LPSA = AE(2) \oplus AO(3). \tag{5.9}$$

The IRs of the algebra $AE(2)$ are of two kinds corresponding to zero and non-zero eigenvalues of the Casimir $C = T_1^2 + T_2^2$. If $C = T_1^2 + T_2^2 = 0$ then

$$T_1 = T_2 = 0, T_0 = \lambda \tag{5.10}$$

where λ is an arbitrary (fixed) integer or half integer. If

$$C = T_1^2 + T_2^2 = r^2 > 0$$

the corresponding IRs are realized by infinite-dimensional matrices. Let $|r, n\rangle$ be the eigenvector of the commuting operators C and T_0, then

$$C \mid r, n\rangle = r^2 \mid r, n\rangle \qquad T_0 \mid r, n\rangle = n \mid r, n\rangle \tag{5.11}$$

$$(T_1 \pm \mathrm{i}T_2) \mid r, n\rangle = r \mid r, n \pm 1\rangle. \tag{5.12}$$

Thus IRs of the algebra (4.4), (5.7) and (5.8) are labelled by pairs of numbers (j, r) (or (j, λ) if $r = 0$). Denoting the common eigenvector of the commuting matrices j^2, j_3, C, T_0 by $|j, \nu; r, n\rangle$ and using (4.8), (5.11) and (5.12) we can represent basis elements of IRs of this algebra in the form

$$\begin{aligned}
&j_3 \mid j, \nu; r, n\rangle = \nu \mid j, \nu; r, n\rangle \qquad \nu = j, j-1, \ldots - j\\
&(j_1 \pm \mathrm{i}j_2) \mid j, \nu; r, n\rangle = \sqrt{j(j+1) - \nu(\nu \pm 1)} \mid j, \nu \pm 1; r, n\rangle\\
&T_0 \mid j, \nu; r, n\rangle = n \mid j, \nu; r, n\rangle\\
&(T_1 \pm \mathrm{i}T_2) \mid j, \nu; r, n\rangle = r \mid j, \nu; r, n \pm 1\rangle\\
&\begin{cases} n = 0, \pm 1, \pm 2, \ldots \text{ or } \quad n = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots, & r \neq 0\\ n = \lambda, & r = 0.\end{cases}
\end{aligned} \tag{5.13}$$

Thus, we have found the explicit form of the IRs of the operators W_ν, Q_A, $\bar{Q}_A$ in the reference frame $P = (M, 0, 0, M)$. To find these operators (and the corresponding generators P_ν, $J_{\nu\sigma}$) in an arbitrary frame of reference it is sufficient to make the corresponding rotation transformation. As a result we obtain

$$\begin{aligned}
&Q_1 = \frac{\sqrt{2}(-p_1 + \mathrm{i}p_2)}{\sqrt{p + p_3}}(j_1 - \mathrm{i}j_2) \qquad \bar{Q}_1 = \frac{\sqrt{2}(-p_1 - \mathrm{i}p_2)}{\sqrt{p + p_3}}(j_1 + \mathrm{i}j_2)\\
&Q_2 = \sqrt{2(p + p_3)}(j_1 - \mathrm{i}j_2) \qquad \bar{Q}_2 = \sqrt{2(p + p_3)}(j_1 + \mathrm{i}j_2)
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&P_0 = \varepsilon p \qquad P_a = p_a\\
&J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc}\hat{T}_0 \frac{p_c + \delta_{c3} p}{p + p_3}\\
&J_{0a} = x_0 p_a - \tfrac{1}{2}\varepsilon[p, x_a]_+ + \frac{\varepsilon_{abc} T_b p_c}{p^2} - \frac{\varepsilon_{abc} p_b n_c (\varepsilon \hat{T}_0 p^2 - T_a p_a)}{p^2 (p + p_3)}
\end{aligned} \tag{5.15}$$

where

$$p = \sqrt{p_1^2 + p_2^2 + p_3^2} \qquad n = (0, 0, 1) \qquad T_3 = 0 \qquad \hat{T}_0 = T_0 - \tfrac{1}{2}(j_3 + j).$$

For the case which is important for physics $C = r^2 = 0$ (representations with discrete spin) formulae (5.15) are simplified and reduced to the form

$$
\begin{aligned}
&P_0 = \varepsilon p \qquad P_a = p_a \\
&J_{ab} = x_a p_b - x_b p_a + \tfrac{1}{2}\varepsilon_{abc}(2\lambda - j - j_3)\frac{p_c + \delta_{c3} p}{p + p_3} \\
&J_{0a} = x_0 p_a - \tfrac{1}{2}\varepsilon[p, x_a]_+ - \tfrac{1}{2}\varepsilon\varepsilon_{abc}(2\lambda - j - j_3)\frac{p_b n_c}{p + p_3}
\end{aligned}
\qquad (5.16)
$$

where λ and j are arbitrary integers or half integers.

So IRs of the PPSA, belonging to class II with $P_0 > 0$, are labelled by the sets of numbers (r, j), $r \neq 0$ or (λ, j) for $r = 0$. The explicit form of the corresponding basis elements is given in (5.14)–(5.16) and (5.13).

6. IRs of class III

To obtain the corresponding LPSA we choose the space-like 4-vector $P = (0, 0, 0, \eta)$. The corresponding double commutation relations (2.2) reduce to the form

$$
\begin{aligned}
&[Q_1, [\bar{Q}_1, Q_B]] = -4\eta Q_B \qquad [\bar{Q}_1, [Q_1, \bar{Q}_B]] = -4\eta \bar{Q}_B \\
&[Q_2, [\bar{Q}_2, Q_B]] = 4\eta Q_B \qquad [\bar{Q}_2, [Q_2, \bar{Q}_B]] = 4\eta \bar{Q}_B
\end{aligned}
\qquad (6.1)
$$

the remaining double commutators are equal to zero. Moreover, denoting

$$
B_0 = -J_{12}\eta + X_0 \equiv \eta \tilde{J}_{12} \qquad B_1 = -J_{02}\eta + X_1 \equiv \eta \tilde{J}_{01} \qquad B_2 = J_{01}\eta + X_2 \equiv \eta \tilde{J}_{02} \qquad (6.2)
$$

and remembering that $B_3 = X_3$, we find from (3.4*b*), that

$$
[\tilde{J}_{\alpha\beta}, Q_A] = [\tilde{J}_{\alpha\beta}, \bar{Q}_A] = 0 \qquad \alpha, \beta = 0, 1, 2 \qquad (6.3)
$$

$$
[\tilde{J}_{\alpha\beta}, \tilde{J}_{\rho\sigma}] = \mathrm{i}(g_{\alpha\sigma}\tilde{J}_{\beta\rho} + g_{\beta\rho}\tilde{J}_{\alpha\sigma} - g_{\alpha\rho}\tilde{J}_{\beta\sigma} - g_{\beta\sigma}\tilde{J}_{\alpha\rho}) \qquad (6.4)
$$

where

$$
g_{00} = -g_{11} = -g_{22} = 1 \qquad g_{\alpha\beta} = 0 \qquad \alpha \neq \beta.
$$

In accordance with (6.1)–(6.4) the LPSA corresponding to space-like momenta reduces to the direct sum of the algebra $AO(1,2)$ (defined by relations (6.4)) and the algebra, defined by the double commutation relations (6.1). The latter reduces to the algebra $AO(2,3)$, if we define Q_A, $\bar{Q}_A$ and the corresponding commutators using the relations (4.9) and (4.10) (with $M \to \eta$, compare (3.6*a*) and (3.6*c*)). Indeed, in this case we immediately find that S_{kl} have to satisfy the algebra $AO(2,3)$. The corresponding commutation relations can be obtained from (4.12) by the change $\delta_{kl} \to -g_{kl}$, where

$$
g_{11} = g_{22} = -g_{33} = -g_{44} = -g_{55} = 1 \qquad g_{kl} = 0 \qquad k \neq l. \qquad (6.5)
$$

Thus we make sure that the LPSA for representations of class III reduces to the direct sum of the algebras $AO(1,2)$ and $AO(2,3)$:

$$
LPSA = AO(1,2) \oplus AO(2,3) \qquad (6.6)
$$

The IRs of the algebra (6.6) can be constructed by analogy with (4.16). For IRs of the algebras $AO(1,2)$ and $AO(2,3)$ see, e.g., [20].

Starting with (6.2), (4.9) and (3.1) and making the Lorentz transformation corresponding to a transition to an arbitrary frame of reference, we find the corresponding basis elements of the PPSA in the form

$$
\begin{aligned}
&P_\mu = p_\mu \qquad J_{ab} = x_a p_b - x_b p_a + \tilde{S}_{ab} \\
&J_{0a} = x_0 p_a - \tfrac{1}{2}[x_a, p_0]_+ + \tilde{S}_{0a} \\
&J_{a3} = x_a p_3 - x_3 p_a - \frac{\tilde{S}_{ab} p_b - \tilde{S}_{a0} p_0}{p_3 + \eta} \\
&J_{03} = x_0 p_3 - \tfrac{1}{2}[x_3, p_0]_+ - \frac{\tilde{S}_{0a} p_a}{p_3 + \eta} \\
&Q_1 = \frac{1}{\sqrt{(\eta + p_3)}}[(\eta + p_3 - p_0)(S_{51} + \mathrm{i}S_{52}) - (p_1 - \mathrm{i}p_2)(S_{53} + \mathrm{i}S_{54})] \\
&Q_2 = \frac{1}{\sqrt{(\eta + p_3)}}[(p_1 + \mathrm{i}p_2)(S_{51} + \mathrm{i}S_{52}) + (\eta + p_3 + p_0)(S_{53} + \mathrm{i}S_{54})] \\
&\bar{Q}_1 = \frac{1}{\sqrt{(\eta + p_3)}}[(\eta + p_3 - p_0)(S_{51} - \mathrm{i}S_{52}) - (p_1 + \mathrm{i}p_2)(S_{53} - \mathrm{i}S_{54})] \\
&\bar{Q}_2 = \frac{1}{\sqrt{(\eta + p_3)}}[(p_1 - \mathrm{i}p_2)(S_{51} - \mathrm{i}S_{52}) + (\eta + p_3 + p_0)(S_{53} - \mathrm{i}S_{54})]
\end{aligned}
\tag{6.7}
$$

where

$$
\begin{aligned}
&p_0^2 = p^2 - \eta^2 \qquad \tilde{S}_{12} = \tilde{J}_{12} + \tfrac{1}{2}(S_{12} + S_{43}) \\
&\tilde{S}_{01} = \tilde{J}_{01} + \tfrac{1}{2}(S_{13} + S_{42}) \qquad \tilde{S}_{02} = \tilde{J}_{02} + \tfrac{1}{2}(S_{32} + S_{41})
\end{aligned}
$$

$\tilde{J}_{\alpha\beta}$ are basis elements of the algebra $AO(1,2)$ (6.4), S_{kl} are basis elements of the algebra $AO(2,3)$ with the metric tensor (6.5), besides $[\tilde{J}_{\alpha\beta}, S_{kl}] = 0$.

7. Covariant representations

Here we present a special realization of representations of the PPSA when the Poincaré group generators have the form

$$P_\mu = p_\mu \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu} \tag{7.1}$$

with $S_{\nu\sigma}$ being numerical matrices. Such a realization (when the 'spin part' $S_{\nu\sigma}$ of generators commutes with 'orbital part' $x_\nu p_\sigma - x_\sigma p_\nu$) can be more revealing in physics than the realizations considered so far.

We choose $S_{\nu\sigma}$ in the form

$$S_{ab} = \varepsilon_{abc} S_c \qquad S_{0a} = -\mathrm{i} S_a \tag{7.2}$$

where S_a are the matrices defined in (4.15). Then the corresponding parasupercharges are

$$
\begin{aligned}
&Q_1 = \sqrt{2M}(-S_{51} + \mathrm{i}S_{52}) \qquad Q_2 = \sqrt{2M}(S_{53} - \mathrm{i}S_{54}) \\
&\bar{Q}_1 = \sqrt{\frac{2}{M}}[(p_3 - p_0)(S_{51} + \mathrm{i}S_{52}) + (p_1 + \mathrm{i}p_2)(S_{53} + \mathrm{i}S_{54})] \\
&\bar{Q}_2 = \sqrt{\frac{2}{M}}[(p_0 + p_3)(S_{53} + \mathrm{i}S_{54}) + (p_1 - \mathrm{i}p_2)(S_{51} + \mathrm{i}S_{52})].
\end{aligned}
\tag{7.3}
$$

To obtain the realizations (7.1)–(7.3) it is sufficient to use the transformation (7.4) and (7.5) given in the following. Moreover, it is easy to verify that the operators (7.1)–(7.3) satisfy the relations (2.1)–(2.3), i.e. realize a representation of this algebra. Besides, if we assume $P_\nu P^\nu = M^2 > 0$, $p_0 = (p^2 + M^2)^{1/2}$, and the matrices $S_{\nu\sigma}$, j_a of (4.15), (7.2) and (7.3) have the form (4.16), then this representation is irreducible and belongs to class I. Indeed, the corresponding operators (7.1)–(7.3) reduce to the form (4.18) and (4.19) using the transformation

$$\begin{aligned} J_{\mu\nu} &\to U J_{\mu\nu} U^{-1} \qquad & P_\mu &\to U P_\mu U^{-1} \\ Q_A &\to U Q_A U^{-1} \qquad & \bar{Q}_A &\to U \bar{Q}_A U^{-1} \end{aligned} \tag{7.4}$$

where

$$U = \exp\left(-\frac{\mathrm{i}S_{0a}p_a}{p}\operatorname{arth}\frac{p}{E}\right). \tag{7.5}$$

8. Discussion

Considering IRs of the PPSA we restricted ourselves to the case of positive values of the Casimir operator $C_3 = P_0/|P_0|$. The case of negative energies can be analysed in complete analogy with the above but corresponds to another LPSA in comparison with (4.14) and (5.9). Moreover, in this case we have $LPSA = AO(3) \oplus AO(1,4)$ for IRs of class I and $LPSA = AE(2) \oplus AO(1,2)$ for IRs of class II. The corresponding parsupercharges can be obtained from (4.18) and (5.14) by the changes $S_{5a} \to \mathrm{i}S_{5a}$, $j_\alpha \to \mathrm{i}j_\alpha$, $a = 1,2,3,4$, $\alpha = 1,2$, where $S_{\mu\nu}$ and j_a are now basis elements of the algebras $AO(1,4)$ and $A0(1,2)$ correspondingly. They satisfy the relations (4.12) with $\delta_{ab} \to -g_{ab}$, where $g_{11} = g_{22} = g_{33} = g_{44} = -g_{55} = -1$ and $g_{11} = g_{22} = -g_{33} = -1$.

Thus, we have described all possible (up to equivalence) IRs of the PPSA. Here we discuss possible physical interpretations of them.

We start with IRs of class I. First, let us discuss the spin contents of these representations. To do this we reduce them to representations of the Poincaré algebra $AP(1,3)$ (which is a subalgebra of the PPSA).

Let us restrict ourselves to the case $j = 0$ (refer to (4.15) and (4.7)). Calculating the corresponding Casimir operator $C = W_\nu W^\nu$ for the subalgebra $AP(1,3)$ we obtain from (4.17) and (4.15)

$$W_\mu W^\mu = M^2 S^2 \qquad S = (S_1, S_2, S_3) \tag{8.1}$$

where

$$S_a = \tfrac{1}{2}(\tfrac{1}{2}\varepsilon_{abc}S_{bc} + S_{4a}) \tag{8.2}$$

and S_{ab}, S_{4a} belong to the IR $D(n_1, n_2)$ of the algebra $AO(5)$.

The matrices (8.2) realize a reducible representation of the algebra $AO(3)$. Indeed, reducing the IR $D(n_1, n_2)$ to the representations of the algebra $AO(4) \supset S_{ab}, S_{4a} (a, b = 1,2,3)$, and continuing this reduction to $AO(3) \supset S_a$ of (8.2), we obtain† the following set of eigenvalues for (8.1)

$$W_\mu W^\mu = -M^2 s(s+1) \qquad s = \frac{n_1+n_2}{2}\ \frac{n_1+n_2-1}{2}\ \frac{n_1+n_2-2}{2}\ \ldots, 0. \tag{8.3}$$

Moreover, the multiplicity M_s of any value of s (i.e., the degeneration of the corresponding eigenvalue $M^2 s(s+1)$ of $W_\nu W^\nu$) is given by the following formulae

$$M_s = \begin{cases} (n_1 - n_2 + 1)(n_1 + n_2 + 1 - 2s) & s \geqslant \frac{n_1 - n_2}{2}, \\ (2n_2 + 1)(2s + 1) & s < \frac{n_1 - n_2}{2}. \end{cases} \tag{8.4}$$

† For the details connected with IRs of the algebras $AO(5) \supset AO(4) \supset AO(3)$ see, e.g., [18].

For the case $j \neq 0$ (see (4.7) and (4.15)) the possible spin values can be found as a result of summation of the two momenta, i.e., j and S of (8.2). As a result we have instead of (8.3)

$$s = \frac{n_1 + n_2}{2} + j \qquad \frac{n_1 + n_2}{2} + j - 1, \ldots s_0 \qquad s_0 = \begin{cases} 0, & \frac{n_1 + n_2}{2} \geqslant j \\ j - \frac{n_1+n_2}{2} & \frac{n_1 + n_2}{2} < j. \end{cases} \tag{8.5}$$

The corresponding multiplicities can be calculated using the Clebsh–Gordon theorem and bearing in mind (8.4).

In accordance with the above IRs of the PPSA can be set into correspondence with parasupermultiplets of particles with spin described by formulae (8.4) and (8.5).

Like supermultiplets [2], parasupermultiplets includes both bosons and fermions.

Let us consider some examples of IRs. For $n_1 = n_2 = 1/2$ we come to IRs of the Poincaré superalgebra. Indeed, in this case the corresponding operators Q_A and $\bar{Q}_A$ of (4.18) satisfy the anticommutation relations (2.4), defining supercharges. Moreover, the related formulae (8.3)–(8.5) reduce to the well known relations (see, e.g., [2])

$$s = j + \tfrac{1}{2}, j, j - \tfrac{1}{2} \qquad M_j = 2 \qquad M_{j\pm 1} = 1 \tag{8.6}$$

(the expressions for M_s follow from (8.4) and the Clebsh–Gordan theorem), giving the spin contents of supermultiplets.

Thus, we have obtained IRs of PSA as a particular (and the simplest) case of our more general problem.

For $n_1 = n_2 = p/2$, $p = 1, 2, \ldots$ formulae (7.1)–(7.3) present the realization of generators of the Poincaré parasupergroup, which is equivalent to that found in [15]. The distinguishing feature of our approach is that we use the explicit matrix constructions (more precisely, IRs of the algebra $AO(5)$) instead of the paraGrassmanian variables and their derivatives applied in [15]. The last, of course, admit matrix realizations and vice versa, our results can be reformulated using the concept of parasuperfield [15].

Consider IRs of class II with discrete spins. The corresponding basis elements are presented in (5.14) and (5.16).

The considered representations are reducible with respect to the subalgebra $AP(1,3)$. Indeed, calculating the additional Casimir operator of the $AP(1,3)$:

$$C = \frac{J_{12}p_3 + J_{31}p_2 + J_{23}p_1}{p} = \lambda - \tfrac{1}{2}j - \tfrac{1}{2}j_3$$

we find that its eigenvalues $\bar{\lambda}$ (associated with helicities of particles) are

$$\bar{\lambda} = \lambda, \lambda - \tfrac{1}{2}, \lambda - 1, \ldots, \lambda - j. \tag{8.7}$$

Thus, the corresponding parasupermultiplet includes $2j + 1$ particles, both bosons and fermions, the helicities of which are given in (8.7).

For $j = 1/2$ we again come to the IRs of PSA which is a particular case of a more general object, i.e., the Poincaré parasuperalgebra.

Using the transformations found in [21] it is possible to find realizations of IRs of the PPSA which are uniform for any class I–III of (3.6a). Such realizations are unitary equivalent to those already considered.

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Appendix

The orthogonal group $O(5)$ is the set of all linear transformations of the five-dimensional Euclidean space preserving the quadratic form $x_1^2 + x_2^2 + \ldots + x_5^2$. The Lie algebra of this group is characterized by relations (4.12). Irreducible representation of the algebra $AO(5)$ are labelled by pairs of numbers n_1 and n_2 (simultaneously integer or half integer).

Each representation of the algebra $AO(5)$ generates a representation of the algebra $AO(4)$. In the Gel'fand–Zetlin basis [18] all the Casimir operators of the subalgebras $AO(4) \supset AO(3) \supset AO(2)$ are diagonal and are characterized by the eigenvalues m_1, m_2, where $n_1 \geqslant m_1 \geqslant n_2 \geqslant m_2 \geqslant -n_2$; l, where $m_1 \geqslant l \geqslant |m_2|$, m, where $l \geqslant m \geqslant -l$, correspondingly.

Numerating basis elements by multi-index

$$\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix}$$

we can represent the action of generators in the form (m_1 and m_2 are fixed)

$$S_{21}\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix} = m\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix}$$

$$S_{32}\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix} = -\frac{\mathrm{i}}{2}\sqrt{(l-m)(l+m+1)}\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m+1 \end{pmatrix} + \frac{\mathrm{i}}{2}\sqrt{(l-m+1)(l+m)} \times\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m-1 \end{pmatrix}$$

$$S_{43}\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix} = \sqrt{\frac{(l+m+1)(l-m+1)(m_1-l)(m_1+l+2)(l-m_2+1)(l+m_2+1)}{(2l+1)(2l+3)(l+1)^2}} \times\xi \begin{pmatrix} m_1\ m_2 \\ l+1 \\ m \end{pmatrix} + \mathrm{i}\frac{m(m_1+1)m_2}{(l+1)l}\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix} - \sqrt{\frac{(l+m)(l-m)(m_1-l+1)(m_1+l+1)(l-m_2)(l+m_2)}{(2l+1)(2l-1)l^2}} \times\xi \begin{pmatrix} m_1\ m_2 \\ l-1 \\ m \end{pmatrix}$$

$$S_{54}\xi \begin{pmatrix} m_1\ m_2 \\ l \\ m \end{pmatrix} = \sqrt{\frac{(m_1-l+1)(m_1+l+2)(n_1-m_1)(n_1+m_1+3)(m_1-n_2+1)(m_1+n_2+2)}{(m_1+m_2+1)(m_1+m_2+2)(m_1-m_2+1)(m_1-m_2+2)}} \times\xi \begin{pmatrix} m_1+1\ m_2 \\ l \\ m \end{pmatrix}$$

$$+\sqrt{\frac{(l-m_2)(m_2+l+1)(n_2-m_2)(n_2+m_2+1)(n_1-m_2+1)(n_1+m_2+2)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2)(m_1-m_2+1)}}$$

$$\times\xi\begin{pmatrix} m_1\ m_2+1 \\ l \\ m \end{pmatrix}$$

$$+\frac{\mathrm{i}}{2}\sqrt{\frac{(m_1+l+1)(m_1-l)(n_1-m_1+1)(n_1+m_1+2)(m_1-n_2)(m_1+n_2+1)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2)(m_1-m_2+1)}}$$

$$\times\xi\begin{pmatrix} m_1-1\ m_2 \\ l \\ m \end{pmatrix}$$

$$+\frac{\mathrm{i}}{2}\sqrt{\frac{(l-m_2+1)(m_2+l)(n_2-m_2+1)(n_2+m_2)(n_1-m_2+2)(m_2+n_1+1)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2+2)(m_1-m_2+1)}}$$

$$\times\xi\begin{pmatrix} m_1\ m_2-1 \\ l \\ m \end{pmatrix}.$$

Other generators can be obtain from (4.12).

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